**LEBANESE AMERICAN UNIVERSITY**
**Division of Computer Science and Mathematics**

# Calculus III

## Exam II

**Fall 2008 (December 11, 2008)**

**Name:** Solutions. **ID:** ____________

**Circle the name of your instructor:** **Dr. Habre** **Dr. Hamdan:**

| Question Number | Grade |
|---|---|
| 1. 8 % | |
| 2. 8% | |
| 3. 24% | |
| 4. 24% | |
| 5. 9% | |
| 6. 27% | |
| Total | |

1. (8%) Examine convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n}$ and determine its sum.

$$= \sum_{1}^{\infty} \left(-\frac{2}{3}\right)^n \quad \text{Geom. series with } r = -\frac{2}{3}$$

$|r|<1 \Rightarrow$ Geom. series converges to:

$$\frac{\text{first term}}{1-r} = \frac{-2/3}{1+2/3} = \frac{-2/3}{5/3}$$

$$= \boxed{-2/5}$$

2. (8%) Consider the series: $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

(a) Find the third and fourth partial sums: $s_3$ and $s_4$.

$$s_3 = a_1 + a_2 + a_3 = \frac{1}{1(3)} + \frac{1}{2(4)} + \frac{1}{3(5)} = \frac{63}{120}$$

$$s_4 = a_1 + a_2 + a_3 + a_4 = s_3 + \frac{1}{4(6)} = \frac{63}{120} + \frac{1}{24}$$

(b) Find the *nth* partial sum $s_n$, and then deduce the sum of the series.

$$s_n = \frac{1}{1(3)} + \frac{1}{2(4)} + \cdots$$

Use Partial Fractions.

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{(A+B)n + 2A}{n(n+2)} \Rightarrow$$

$A+B=0$ $\quad A = 1/2 \quad B = -1/2$.

$2A = 1$

$$\Rightarrow \sum_{1}^{\infty} \left(\frac{1/2}{n} - \frac{1/2}{n+2}\right)$$

$$s_n = \frac{1}{2}\Bigg(\underbrace{\left(1-\frac{1}{3}\right)}_{a_1} + \underbrace{\left(\frac{1}{2}-\frac{1}{4}\right)}_{a_2} + \underbrace{\left(\frac{1}{3}-\frac{1}{5}\right)}_{a_3} + \underbrace{\left(\frac{1}{4}-\frac{1}{6}\right)}_{a_4} + \cdots + \underbrace{\left(\frac{1}{n}-\frac{1}{n+2}\right)}_{a_n}\Bigg)$$

$$\text{Sum} = \frac{1}{2}\left(1+\frac{1}{2}\right) = \boxed{3/4}$$

3. (24%) Determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ Ratio Test

$$\rho = \lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$\rightarrow \frac{(n+1)\, n^n}{(n+1)(n+1)^n} = \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \longrightarrow \frac{1}{e}$$

$\rho = \frac{1}{e} < 1 \Rightarrow$ Series Conv. by Ratio Test.

(b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

$$\frac{\ln n}{n^2} < \frac{n^{0.5}}{n^2} = \frac{1}{n^{1.5}} \Rightarrow$$ Converges by DCT

Since $\sum \frac{1}{n^{1.5}}$ (p-series $p > 1$)

(c) $\sum_{n=1}^{\infty} \frac{n}{e^n + 5}$ $\qquad \frac{n}{e^n + 5} < \frac{1}{n^2}$ Cross Multiply.

$\Rightarrow$ Conv. by DCT

since $\sum \frac{1}{n^2}$ conv. $(p > 1)$

4. (24%) Determine whether the following series converge absolutely, conditionally, or diverge:

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+2n+2}$

$\sum \frac{1}{n^2+2n+2} \approx \sum \frac{1}{n^2}$ Conv. by LCT

(since $\sum \frac{1}{n^2}$ $p=2>1$ conv).

$\therefore$ The original series converges Absolutely.

(b) $\sum_{n=1}^{\infty}(-1)^n \left(1-\frac{1}{n}\right)^{2n}$

$a_n = \left(1-\frac{1}{n}\right)^{2n} \longrightarrow \left(e^{-1}\right)^2$

$= e^{-2}$

$a_n \not\to 0$. $\therefore$ Series diverges by nth term test.

(c) $\sum_{n=2}^{\infty}(-1)^n \left(\frac{1}{n^{1/3}+7}\right)$

1 Check $\sum \frac{1}{n^{1/3}+7} \approx \sum \frac{1}{n^{1/3}}$ diverges p-series $p<1$

$\Rightarrow$ Series does not converge absolutely

2 Leibnitz? $a_n \searrow 0$ $\therefore$ Series conv. Conditionally

5. (9%) Find the values of $x$ for which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$ converges: Explain.

$$\rho = \lim \frac{1}{n^{1/n}} \left|\frac{x}{3}\right| \longrightarrow \left|\frac{x}{3}\right|.$$

Need $x$ so that $\rho < 1 \Rightarrow \left|\frac{x}{3}\right| < 1 \Rightarrow$

$-3 < x < 3$

Check end points.

(1) $x = -3 \Rightarrow$ Series yields: $\sum \frac{(-1)^n}{n}$ conv. cond.

(2) $x = 3 \Rightarrow \sum \frac{1}{n} \Rightarrow$ series div.

Conclusion: We need $\boxed{-3 \leq x < 3}$

6. (27%) Using Maclaurin series, answer the following questions:

(a) Find $\lim_{x\to\infty} x^2(e^{10/x^2} - 1)$

$$\lim_{x\to\infty} x^2\left(1 + \frac{10}{x^2} + \frac{(10/x^2)^2}{2!} + \frac{(10/x^2)^3}{3!} + \cdots - 1\right)$$

$$= \lim_{x\to\infty} \left(x^2\left(\frac{10}{x^2} + \frac{10^2}{2!\,x^4} + \frac{10^3}{3!\,x^6} + \cdots\right)\right)$$

$$= \lim_{x\to\infty} 10 + \underbrace{\frac{10^2}{2!\,x^2}}_{\to 0} + \cdots$$

$$= \boxed{10}$$

(b) Represent the function $f(x) = \frac{x^2}{1-6x}$ as a power series

$$x^2\left(\frac{1}{1-6x}\right) = x^2\left(1+6x+(6x)^2+(6x)^3+(6x)^4+\cdots\right).$$

Used: $\frac{1}{1-6x} = \frac{1}{1-r} = 1+r+r^2+\cdots$ where

$|r| = |6x| < 1$

Need $|x| < 1/6$

$$f(x) = x^2 + 6x^3 + 6^2x^4 + 6^3x^5 + 6^4x^6 + \cdots$$

$$\therefore \text{ function } = \sum_{n=0}^{\infty} 6^n x^{n+2}$$

(c) Write the integral $\int e^{-x^2 dx}$ as a power series.

$$\int e^{-x^2}dx = \int 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{2!} - \frac{x^7}{7(3!)} + \frac{x^9}{9(4!)} - \cdots$$